

Stochastic duality of Markov processes: a study via generators

Vassili Kolokoltsov and RuiXin Lee

April 8, 2013

Abstract

The paper is devoted to a systematic study of the duality of processes in the sense that $Ef(X_t^x, y) = Ef(x, Y_t^y)$ for a certain f . This classical topic has well known applications in interacting particles, intertwining, superprocesses, stochastic monotonicity, exit - entrance laws, ruin probabilities in finances, etc. Aiming mostly at the case of f depending on the difference of its arguments, we shall give a systematic study of duality via the analysis of the generators of dual Markov processes leading to various results and insights.

Key words: stochastic monotonicity, stochastic duality, generators of dual Markov processes, reflection and absorbtion

1 Introduction

The paper is devoted to a systematic study of the duality of processes in the sense that $Ef(X_t^x, y) = Ef(x, Y_t^y)$ for a certain f . This classical topic has well known applications in (and deep links with) interacting particles (see e.g. [25] and references therein), intertwining (see e.g. [7], [8], [13], [27]), superprocesses (see [15], [26]), stochastic monotonicity (see e.g. [32] and [10]), exit - entrance laws (see [11]), ruin probabilities in finances (see [12]), birth and death processes (see [31], [2]) and others. Aiming mostly at the case of f depending on the difference of its arguments, we shall give a systematic study of duality via the analysis of the generators of dual Markov processes (extending the analysis of one-dimensional processes from [22], [21]) leading to various results and insights.

1.1 Objectives

In stochastic analysis one meets various kinds of duality. For instance, the Markov processes X_t^x and Y_t^y with values in a Borel space X are called dual with respect to the reference measure ν on X , if the duality equation

$$\int_X \mathbf{E}h(X_t^x)g(x)\nu(dx) = \int_X h(x)\mathbf{E}g(Y_x^t)\nu(dx) \quad (1)$$

holds for an appropriate class of functions h, g , see e.g. [3] and references therein for.

In another approach, the Markov processes X_t^x and Y_t^y (small x, y here and in what follows stand for the initial points) with values in possibly different Borel spaces X and Y are called dual with respect to a Borel function f on $X \times Y$, if

$$\mathbf{E}f(X_t^x, y) = \mathbf{E}f(x, Y_t^y) \quad (2)$$

for all $x \in X, y \in Y$, where \mathbf{E} on the left hand side and the right hand side correspond to the distributions of processes X_t^x and Y_t^y respectively, see e.g. [25] and references therein for an extensive application of this notion in interacting particles.

A particular case of (2) is the duality of one-dimensional processes (X and Y are real-valued) arising from stochastic monotonicity, where $f(x, y) = \mathbf{1}_{\{x \geq y\}}$ (we denote here and in what follows by $\mathbf{1}_M$ the indicator function of the set M) and hence (2) turns to the equation

$$\mathbf{P}(X_t^x \geq y) = \mathbf{P}(Y_t^y \leq x), \quad (3)$$

see [29]. Other useful cases include $f(x, y) = e^{xy}$ or $f(x, y) = x^y$, used in particular in the theory of superprocesses, see e.g. Ch. 4 of [15] or Ch. 1 of [14]. For an application of duality in actuarial science see [12].

The analytic analogs of the duality of the 1st kind is successfully used in the theory of operator semigroups independently of their probabilistic content, see e.g [3] and references therein. We shall start now with a sketch of a systematic study of duality obtained by extending (2) to general purely analytic setting aiming at the extension of the theory of (3) to dualities generated by partial orders and more general translation invariant dualities arising from f depending on the difference of their arguments.

There are many applications of duality in population dynamics, branching processes and other areas, see e.g. [1] and references therein.

1.2 On the general notion of semigroup duality

For a topological (e.g. metric) space X we denote by $B(X)$ and $C(X)$ the spaces of bounded Borel measurable and bounded continuous functions respectively. Equipped with the sup-norm $\|f\| = \sup_x |f(x)|$ both these spaces become Banach spaces. Bounded signed measures on X are defined as bounded σ -additive functions on the Borel subsets of X . The set of such measures $\mathcal{M}(X)$ equipped with the total variation norm is also a Banach space. The standard duality between $B(X)$ and $\mathcal{M}(X)$ is given by the integration:

$$(f, \mu) = \int_X f(x)\mu(dx).$$

Let X, Y be two topological spaces. By a signed (stochastic) kernel from X to Y we mean a function of two variables $p(x, A)$, where $x \in X$ and A are Borel subsets of Y such that $p(x, .)$ is a bounded signed measure on Y for any x and $p(., A)$ is a Borel function for any Borel set A . We say that this kernel is bounded if $\sup_x \|p(x, .)\| < \infty$. We say that this kernel is weakly continuous if the mapping $x \mapsto p(x, .)$ is continuous with measures $\mathcal{M}(Y)$ considered in their weak topology. If all measures $p(x, .)$ are positive, the corresponding kernel is called a stochastic kernel.

Any bounded kernel specifies a bounded linear operator $B(Y) \rightarrow B(X)$ via the formula

$$Tg(x) = \int_Y g(z)p(x, dz).$$

T is said to be the integral operator with the kernel p . The standard dual operator T' is defined as the operator $\mathcal{M}(X) \rightarrow \mathcal{M}(Y)$ specified by the duality relation

$$(f, T'\mu) = (Tf, \mu),$$

or explicitly as

$$T'\mu(dy) = \int_X p(x, dy)\mu(dx).$$

Clearly the kernel $p(x, dz)$ is weakly continuous if and only if T acts on continuous functions, that is, $T : C(Y) \rightarrow C(X)$.

Let $p(x, dz)$ be a bounded signed kernel from X to itself, T the corresponding integral operator, and let $f(x, y)$ be a bounded measurable function on $X \times Y$. Let us say that the operator $T^{D(f)} : B(Y) \rightarrow B(T)$ is f -dual to T , if

$$(T^{D(f)}f(x, .))(y) = (Tf(., y))(x) \quad (4)$$

for any x, y , that is, the application of T^D to the second argument of f is equivalent to the application of T to its first argument. Of course, if $T^{D(f)}$ is f -dual to T , then T is \tilde{f} -dual to $T^{D(f)}$ with $\tilde{f}(y, x) = f(x, y)$.

We say that f separates points of X if, for any $x_1, x_2 \in X$, there exists $y \in Y$ such that $f(x_1, y) \neq f(x_2, y)$. The following is a bit more nontrivial notion. We say that f separates measures on X if, for any $Q_1, Q_2 \in \mathcal{M}(X)$, there exists $y \in Y$ such that $\int f(x, y)Q_1(dx) \neq \int f(x, y)Q_2(dx)$. If this is the case, the integral operator $F = F_f : \mathcal{M}(X) \rightarrow B(Y)$ given by

$$(FQ)(y) = \int f(x, y)Q(dx) \quad (5)$$

is an injective bounded operator, so that the linear inverse F^{-1} is defined on the image $F(\mathcal{M}(X))$. Let us say that the function FQ is f -generated by Q .

Remark 1. In [11] the authors call a function g to be representable by f , if there exists a unique Q such that $g = FQ$. Paper [11] deals with the application of duality to exit and entrance laws of Markov processes.

1.3 Basic tools

Proposition 1.1. Let f be a bounded measurable function separating measures on X and T an integral operator in $B(X)$ with a bounded signed kernel p . Then $T^{D(f)}$ is well defined on $F(\mathcal{M}(X))$ and its action on the f -generated functions coincides with T' , that is

$$T^{D(f)} = F \circ T' \circ F^{-1}, \quad (6)$$

or equivalently

$$F^{-1} \circ T^{D(f)} = T' \circ F^{-1}. \quad (7)$$

In other words, the f -dual operator $T^{D(f)}$ is obtained by the 'dressing' of the standard dual T' by the operator F .

Proof. Let $g \in F(\mathcal{M}(X))$ be given by $g(y) = \int f(x, y)Q_g(dx)$. Then

$$\begin{aligned} T^{D(f)}g(y) &= \int_X (T^{D(f)}f(x, .))(y)Q_g(dx) \\ &= \int_X (Tf(., y))(x)Q_g(dx) = \int_X \int_Y f(z, y)p(x, dz)Q_g(dx) = \int_Y f(z, y)\tilde{Q}(dz), \end{aligned}$$

with

$$\tilde{Q}(dz) = \int p(x, dz)Q_g(dx).$$

Thus $T^{D(f)}g$ is f -generated by $\tilde{Q} = T'Q_g$, as required. \square

Remark 2. Equation (7) is a particular case of the so-called intertwining, see [7], [8], as well as [13], [27], [18] for exciting recent developments. Relations (7) for discrete Markov chains are analyzed in detail in [19].

Representation (6) has a direct implication for the theory of semigroups.

Proposition 1.2. Let f be a bounded measurable function separating measures on X and T_t a semigroup of integral operators in $B(X)$ specified by the family of bounded signed kernel $p_t(x, dz)$ from X to X , so that T_0 is the identity operator and $T_tT_s = T_{t+s}$, which, in terms of kernels, rewrites as the Chapman-Kolmogorov equation

$$\int_X p_t(x, dz)p_s(z, dw) = p_{t+s}(x, dw).$$

Then the dual operators $T_t^{D(f)}$ in $F(\mathcal{M}(X))$ also form a semigroup, so that

$$T_t^{D(f)} = F \circ T'_t \circ F^{-1}. \quad (8)$$

Proof. This is straightforward from (6) and the standard obvious fact that T'_t form a semigroup in $\mathcal{M}(X)$. \square

Remark 3. The duality (1) is of course also included in the general scheme above, that is, the dual can still be expressed as (6). For instance, if $\nu(dx)$ has a density $\nu(x)$ with respect to Lebesgue measure and T' can be reduced to the action on functions, then F^{-1} is the multiplication on $n\nu(x)$ and $f(x, y) = \delta(x - y)\nu^{-1}(x)$.

It is also worth noting that the assumption of boundedness of f is not very essential. If it is not bounded (and we shall discuss interesting examples of such situations later), the integral operator F will not be defined on all bounded measures, but only on its subspace. This will be reflected in the domain of $T^{D(f)}$, but the whole scheme of Proposition 1.1 still remains valid.

1.4 Links with differential equations and stochastic processes

Let us explain briefly the main ideas on the application of the above results to the theory of differential equations and stochastic processes. Precise details for concrete situations will be discussed below.

Let a semigroup T_t in $B(X)$ be generated by a (possibly unbounded) operator L in $B(X)$ defined on an invariant (under all T_t) domain $D \subset B(X)$, so that

$$\frac{d}{dt} \Big|_{t=0} T_t h = \lim_{t \rightarrow 0} \frac{1}{t} (T_t h - h) = Lh, \quad h \in D,$$

with convergence in some appropriate topology (say, strongly or point-wise) and thus the operators T_t represent resolving operators for the Cauchy problem of the equation $\dot{h} = Lh$. Then (6) implies that

$$\frac{d}{dt} \Big|_{t=0} T_t^{D(f)} g = F \circ \frac{d}{dt} \Big|_{t=0} T' \circ F^{-1} g = F \circ L' \circ F^{-1} g,$$

that is, the generator of the semigroup $T_t^{D(f)}$ is

$$L^{D(f)} = F \circ L' \circ F^{-1}, \tag{9}$$

so that the operators $T_t^{D(f)}$ represent resolving operators for the Cauchy problem of the equation $\dot{g} = L^{D(f)} g$. Here L' is of course the standard dual operator to L . Thus duality can yield explicit solutions for equations of this kind. Of course, our arguments were heuristic as we did not pay attention to the domain of definition of L' , which should be done in concrete situations. The main difficulty here is to characterize the operator F_f .

Next, in order to be able to fill the duality equation (4) with probabilistic content, i.e. to rewrite it as (2), the semigroups T_t and $T_T^{D(f)}$ should be positivity preserving and generate some Markov processes.

This question effectively reduces to the question of whether, for a given conditionally positive operator L , the corresponding dual $L^{D(f)}$ is also conditionally positive.

It is seen now that the basic questions to be addressed to make the theory work for concrete functions f are (i) the characterization of the operators F and F^{-1} (for the analytic part of the story) and (ii) the criteria for conditional positivity of $L^{D(f)}$ (for its probabilistic content).

As we shall see it is often convenient to reduce the operator F to some subclass of Borel measures Q , where its inverse can be explicitly found. For instance, it is often easier to work with Q having density with respect to some reference measure.

1.5 Content and plan of the paper

We shall apply formulas (8) and (9) to characterize classes of dual Markov processes with respect to various functions f depending on the difference of its arguments. Section 2 deals with duality on \mathbf{R}^d arising from Pareto and similar partial orders. The full characterization of duality is given in terms of generators for basic classes of Feller processes. Section 3 discusses several examples of duality with operator F^{-1} being the Laplacian or a fractional Lapacian. Section 4 initiates an application of formulas (8) and (9) to the study of duality for processes in domains with a boundary. To circumvent specific difficulties arising from the boundary, we introduce here an additional tool of a regularized dual.

The extension of the theory to time-nonhomogeneous Markov processes will be analyzed in [24].

2 Duality from orders and other binary relations

2.1 Basic notions

As our basic example we consider f -duality for functions f arising from translation-invariant partial orders, or more generally, from translation-invariant binary relations. Namely, let X be a topological linear space and M a Borel subset of X . Then M defines a translation-invariant binary relation R_M on X such that xR_My means, by definition, that $x - y \in M$, or $x \in y + M$.

Let $\tilde{M} = \{(x, y) \in X \times X : xR_My\}$. Let us say that the duality (4) arises from the binary relation M , if

$$f(x, y) = f_M(x, y) = \mathbf{1}_{\tilde{M}}(x, y) = \mathbf{1}_{x-M}(y) = \mathbf{1}_{y+M}(x). \quad (10)$$

Remark 4. If f -duality arises from a translation-invariant binary relation R_M and if both T_t and $T_t^{D(f)}$ are known to be integral operators with kernels $p_t(x, dz)$ and $p_t^{D(f)}(y, dw)$ respectively, one can give another instructive proof of Proposition 1.2 bypassing representation (6) and using instead Fubini's theorem, as was done in [29] for standard one-dimensional duality. Namely, it is sufficient to show the semigroup identity $T_{t+s}^{D(f)} = T_s^{D(f)}T_t^{D(f)}$ applied to the functions $f(x, .) = \mathbf{1}_{x-M}$, as it then extends to the whole $F(\mathcal{M}(X))$ by linearity. And for these functions we have

$$\begin{aligned} (T_{t+s}^{D(f)}\mathbf{1}_{x-M})(y) &= (T_{t+s}\mathbf{1}_{y+M})(x) = (T_t(T_s\mathbf{1}_{y+M}))(x) = \int p_t(x, dz)(T_s\mathbf{1}_{y+M})(z) \\ &= \int p_t(x, dz)(T_s^{D(f)}\mathbf{1}_{z-M})(y) = \int p_t(x, dz) \left(\int \mathbf{1}_{z-M}(w)p_s^{D(f)}(y, dw) \right). \end{aligned}$$

Applying Fubini's theorem this rewrites as

$$\int p_s^{D(f)}(y, dw) \int \mathbf{1}_{w+M}(z)p_t(x, dz) = \int (T_t\mathbf{1}_{w+M})(x)p_s^{D(f)}(y, dw) = T_s^{D(f)}(T_t^{D(f)}\mathbf{1}_{x-M})(y),$$

as required.

If M contains the origin and is closed under the addition of vectors, then the relation R_M is a pre-order (i.e. it is reflexive and transient) and can be naturally denoted by \geq_M . If this is the case and T_t and $T_t^{D(f)}$ are integral operators with positive stochastic kernels thus specifying Markov processes, then duality relation (4) or equivalently (2) turns to the equation

$$\mathbf{P}(X_t^x \geq_M y) = \mathbf{P}(Y_t^y \leq_M x), \quad (11)$$

extending one-dimensional duality (3).

The basic example we are going to analyze now is the Pareto partial order in $X = \mathbf{R}^d$, i.e. \geq_M with $M = \mathbf{R}_+^d$, and its natural extension with $M = C(e_1, \dots, e_d)$ the cone generated by d linear independent vectors $\{e_1, \dots, e_d\}$ in \mathbf{R}^d :

$$C(e_1, \dots, e_d) = \{x = \sum_{j=1}^d \alpha_j e_j : \alpha_j \geq 0, j = 1, \dots, d\}. \quad (12)$$

Of course the relation \geq_M with such M is again a Pareto order, but in a transformed system of coordinates.

Let us start with $M = \mathbf{R}_+^d$ corresponding to the Pareto order, which we shall denote just by \geq omitting the subscript M . The corresponding dual semigroups or processes (if exist) will be referred to as Pareto dual. In this case

$$(FQ)(y) = \int f_M(x, y)Q(dx) = \int_{x \geq y} Q(dx) \quad (13)$$

is just the usual multidimensional distribution function for the measure Q on \mathbf{R}^d . It is known (and easy to see) that FQ characterizes Q uniquely implying that F is injective and thus f_M separates measures on \mathbf{R}^d yielding the main condition of Proposition 1.1. Moreover, if Q has a density q with respect to Lebesgue measure, then q can be found from $FQ = g$ by differentiation:

$$q(y_1, \dots, y_d) = F^{-1}g(y) = (-1)^d \frac{\partial^d g(y)}{\partial y_1 \dots \partial y_d}. \quad (14)$$

Thus, for the Pareto order, the operator F^{-1} has the simple explicit expression.

In the case of the orders arising from the cones $M = C(e_1, \dots, e_d)$ given by (12) this formula generalizes to

$$q(y_1, \dots, y_d) = (F^{-1}g)(y) = (-1)^d \frac{\frac{\partial^d g}{\partial y^d}(y)[e_1, e_2, \dots, e_d]}{|\det(e_1, e_2, \dots, e_d)|}, \quad (15)$$

where $\det(e_1, e_2, \dots, e_d) = \det(e_i^j)$ is the determinant of the matrix whose i th columns consist of the coordinates of the vector e_i and

$$\frac{\partial^d g}{\partial y^d}(y)[e_1, e_2, \dots, e_d] = \sum_{i_1, i_2, \dots, i_d} \frac{\partial^d g}{\partial y_{i_1} \dots \partial y_{i_d}}(y) e_1^{i_1} e_2^{i_2} \dots e_d^{i_d}.$$

Remark 5. For completeness, let us sketch a proof of this formula. If a measure Q on \mathbf{R}^d has a continuous density q , so that

$$g(x) = FQ(x) = \int_{y+C(e_1, \dots, e_d)} q(z)dz,$$

the function q can be clearly found as the limit

$$q(y) = \lim_{h \rightarrow 0} \int_{y+\Pi(he_1, \dots, he_d)} q(z) dz |\Pi(he_1, \dots, he_d)|^{-1}, \quad (16)$$

where

$$\Pi(he_1, \dots, he_d) = \{x = \sum_j \alpha_j he_j, \quad \alpha_j \in [0, 1]\}$$

is the parallelepiped built on the vectors $\{he_1, \dots, he_d\}$ and

$$|\Pi(he_1, \dots, he_d)| = h^d |\det(e_i^j)|$$

is its Euclidean volume.

From simple combinatorics it follows (see e.g. [20]) that

$$\begin{aligned} & \int_{y+\Pi(h e_1, \dots, h e_d)} q(z) dz \\ = g(y) - \sum_j g(y + h e_j) + \sum_{i < j} g(y + h e_i + h e_j) + \dots + (-1)^d g(y + h e_1 + \dots + h e_d). \end{aligned}$$

Let us expand all terms in Taylor series up to the derivatives of order d . As the final expression should be of order h^d (to get a limit in (16)) we conclude that all terms with the derivatives of orders less than d necessarily cancel, so that

$$\begin{aligned} & \int_{y+\Pi(h e_1, \dots, h e_d)} q(z) dz \\ = \frac{1}{d!} h^d & \left(- \sum_j \frac{\partial^d g}{\partial y^d}[e_j] + \sum_{i < j} \frac{\partial^d g}{\partial y^d}[e_i + e_j] + \dots + (-1)^d \frac{\partial^d g}{\partial y^d}[e_1 + \dots + e_d] \right) + O(h^{d+1}), \end{aligned} \quad (17)$$

where $O(h^{d+1})$ denotes the expression of order h^{d+1} that does not contribute to the limit in (16), and where we use the well established (though a bit ambiguous) notation for the action of the higher order derivative on equal vectors:

$$\frac{\partial^d g}{\partial y^d}(y)[v] = \frac{\partial^d g}{\partial y^d}(y)[v, \dots, v].$$

It remains to note that all terms in expansion (17) containing products of coordinates of coinciding vectors should vanish (otherwise, using different scaling on e_i we would arrive to a contradiction with the existence of the limit in (16)). The only non-vanishing terms should contain the products of d coordinates of all d vectors. All these products comes from the last term in the sum (17) leading to (15).

For instance, let us consider a 'two-dimensional light cone'

$$C(e_1, e_2) = \{(x, y) : y \geq |x|\} \in \mathbf{R}^2, \quad (18)$$

corresponding to vectors $e_1 = (1, 1), e_2 = (-1, 1)$. Then formula (15) for the inverse operator turns to the simple wave operator

$$q(x, y) = F^{-1}g(x, y) = \frac{1}{2} \left(\frac{\partial^2 g}{\partial y^2} - \frac{\partial^2 g}{\partial x^2} \right) (x, y). \quad (19)$$

2.2 Duality from Pareto order: global analysis

Let us now make the detailed analysis of the duality arising from the standard Pareto order in \mathbf{R}^d , i.e. with $M = \mathbf{R}_+^d$. We aim at (i) finding explicitly the dual operator $L^{D(f)}$ for the main classes of the generators of Feller processes in \mathbf{R}^d including diffusions and jump processes and (ii) establishing criteria (in terms of the initial operator L) ensuring that this dual operator is conditionally positive and specifies a Markov process, so that the duality relation (11) holds that we shall write simply as

$$\mathbf{P}(X_t^x \geq y) = \mathbf{P}(Y_t^y \leq x) \quad (20)$$

for the case of the Pareto partial order.

Let us analyze formula (6) from Proposition 1.1. In the case of duality arising from Pareto order and the operator T being integral with a probability kernel $p(x, dz)$ (i.e. all measures $p(x, \cdot)$ are probability measures, as is the case for transition operators of Markov processes) it states that for a distribution function g of a measure Q on \mathbf{R}^d . i.e. $g(x) = \int_{z \geq x} Q(dz)$ we have

$$T^{D(f)} g(x) = F \circ T' \circ F^{-1} g(x) = \int_{y \geq x} \int_{\mathbf{R}^d} p(z, dy) Q(dz). \quad (21)$$

We are interested in the question of when this operator can be extended to all bounded measurable g as a positive operator preserving constants, i.e. as an integral operator with a probability kernel.

Assume first that the measure Q has a continuous density q so that (14) holds, i.e.

$$q(x) = (-1)^d \frac{\partial g^d}{\partial x_1 \cdots \partial x_d}.$$

In this case

$$T^{D(f)} g(x) = (-1)^d \int_{y \geq x} \int_{\mathbf{R}^d} p(z, dy) \frac{\partial g^d}{\partial z_1 \cdots \partial z_d} dz. \quad (22)$$

We like to get rid of the derivatives of g . To be able to do it, let us assume that the kernel $p(x, dz)$ is weakly continuous and has weakly continuous mixed derivatives, that is, for any $I \subset \{1, \dots, d\}$ (including $\{1, \dots, d\}$ itself) the mixed derivative

$$\frac{\partial p^{|I|}}{\partial z_I}(z, dy) \quad (23)$$

is a well defined weakly continuous kernel (possibly signed). Then, integrating the integral over z in (22) by parts d times and assuming that all boundary terms vanish, we get

$$T^{D(f)} g(x) = \int_{\mathbf{R}^d} \left(g(z) \int_{y \geq x} \frac{\partial p^d}{\partial z_1 \cdots \partial z_d}(z, dy) \right) dz. \quad (24)$$

This is an integral operator with the integral kernel (more precisely its density)

$$p^D(x, z) = \int_{y \geq x} \frac{\partial p^d}{\partial z_1 \cdots \partial z_d}(z, dy).$$

For this operator to be positive and constant preserving, necessary conditions are that, for all $x \in \mathbf{R}^d$,

$$\int_{y \geq x} \frac{\partial p^d}{\partial z_1 \cdots \partial z_d}(z, dy) \geq 0, \quad (25)$$

$$\int \left(\int_{y \geq x} \frac{\partial p^d}{\partial z_1 \cdots \partial z_d}(z, dy) \right) dz = 1. \quad (26)$$

From the integration by parts it is seen that for the last condition to hold it is sufficient to assume that for any subset $I \subset \{1, \dots, d\}$ excluding the whole set $\{1, \dots, d\}$,

$$\lim_{z_{\bar{I}} \rightarrow -\infty} \int_{\mathbf{R}^{|I|}} dz_I \int_{y \geq x} \frac{\partial p^{|I|}}{\partial z_I}(z_I, z_{\bar{I}}, dy) = 0, \quad (27)$$

and there exists a finite limit

$$\lim_{z_I \rightarrow \infty} \int_{\mathbf{R}^{|I|}} dz_I \int_{y \geq x} \frac{\partial p^{|I|}}{\partial z_I}(z_I, z_I, dy), \quad (28)$$

which equals 1 for the empty set I . Moreover, one sees by inspection that this condition also ensures that integrating by parts (22) for a g having finite density (14), all boundary terms will in fact vanish, justifying equation (24).

Thus we have proved the following statement.

Proposition 2.1. *Suppose an integral operator T in $B(\mathbf{R}^d)$ is given by a probability kernel $p(x, dy)$ having all mixed derivatives (23) well defined and weakly continuous and such that (25) holds, (27) holds for any subset $I \subset \{1, \dots, d\}$ excluding the whole set $\{1, \dots, d\}$, and there exists a finite limit (28), which equals 1 for the empty set I . Then the Pareto dual operator $T^{D(f)}$ is also an integral operator with a probability kernel.*

Condition (25) is of course not directly verifiable. Therefore we shall see how it can be read from the generator of the process.

2.3 Duality from Pareto order: deterministic and diffusion processes

We plan now to find the generators of the dual processes, when they exist. Let us start with the simplest case of deterministic processes generated by the first order differential operators of the form

$$L\phi(x) = (b(x), \nabla\phi(x)) = \sum_{j=1}^d b_j(x) \frac{\partial\phi}{\partial x_j}. \quad (29)$$

In this case the dual operator is well defined on functions and

$$L'g(x) = -\text{div}(gb)(x) = -\sum_j \frac{\partial}{\partial x_j}[b_j(x)g(x)].$$

For a vector $x = (x_1, \dots, x_d) \in \mathbf{R}^d$ let us denote \check{x}_i the vector in \mathbf{R}^{d-1} obtained from x by deleting the coordinate x_i . For a function $g(x)$ let us write $g(\check{z}_i, x_i)$ for the value of g on the vector, whose i th coordinate is x_i , and other coordinates are those of the vector z . Let us write $d\check{z}_j$ for the product of differentials dz_k with all $k = 1, \dots, d$ excluding j .

Integrating by parts and assuming that g decays quickly enough so that the boundary terms at infinity vanish, we have

$$\begin{aligned} L^{D(f)}g(x) &= FL'F^{-1}g(x) = (-1)^{d+1} \int_{z \geq x} \sum_j \frac{\partial}{\partial z_j} \left[b_j(z) \frac{\partial^d g(z)}{\partial z_1 \cdots \partial z_d} \right] dz_1 \cdots dz_d \\ &= (-1)^d \sum_j \int_{\check{z}_j \geq \check{x}_j} b_j(\check{z}_j, x_j) \frac{\partial^d g(z)}{\partial z_1 \cdots \partial z_d}(\check{z}_j, x_j) d\check{z}_j. \end{aligned} \quad (30)$$

In general one cannot simplify this expression much further, and this is not a conditionally positive operator (it does not have a Lévy-Khintchin form with variable coefficients) without further assumptions.

Proposition 2.2. Let L have form (29) with all $b_j \in C^1(\mathbf{R}^d)$ (the space of bounded continuous functions with bounded continuous derivatives). Then $L^{D(f)}$ is given by (30), so that the solution to the Cauchy problem of the equation $\dot{g} = L^{D(f)}g$ is given by the corresponding formula (6) with F and F^{-1} given by (13) and (14). Moreover, if each b_j depends only on the coordinate x_j , then

$$L^{D(f)}g(x) = -b_j(x_j) \frac{\partial g}{\partial x_j}, \quad (31)$$

that is, $L^{D(f)}$ coincides with L up to a sign and the dual process exists and is just the deterministic motion in the opposite direction to the original one.

Proof. Formula (31) is straightforward from (30) and the assumptions made on b_j . This makes the last statement plausible. However, strictly speaking, having the generator calculated on some subclass of functions does not directly imply that the semigroup $T^{D(f)}$ coincides with the semigroups on $C(\mathbf{R}^d)$ generated by operator (31). The simplest way to see that this is in fact the case is via direct calculations with the semigroup $T_t^{D(f)}$ itself. Namely, if the deterministic Markov process X_t^x with generator (29) can be expressed as $X_t^x = X^t(x)$ via the solutions $X^t(x)$ of the Cauchy problem for the ODE $\dot{x} = b(x)$, its transition kernel takes the form $p_t(z, dy) = \delta(y - X^t(z))$. Then (22) becomes

$$T_t^{D(f)}g(x) = (-1)^d \int_{X^t(z) \geq x} \frac{\partial g^d}{\partial z_1 \cdots \partial z_d} dz. \quad (32)$$

Under the assumption that b_i depend only on x_i , the coordinates of $X^t(z)$ are themselves solutions $X_i^t(z_i)$ of the one-dimensional ODE $\dot{x}_i = b_i(x_i)$, so that one has

$$T_t^{D(f)}g(x) = (-1)^d \int_{X_i^t(z_i) \geq x_i} \frac{\partial g^d}{\partial z_1 \cdots \partial z_d} dz. \quad (33)$$

From the obvious monotonicity of one-dimensional ODE this rewrites as

$$T^{D(f)}g(x) = (-1)^d \int_{z_i \geq (X_i^t)^{-1}(x_i)} \frac{\partial g^d}{\partial z_1 \cdots \partial z_d} dz = g((X^t)^{-1}(x)), \quad (34)$$

which is of course the semigroup generated by the operator (31). \square

Let us turn to a diffusion operator having the form

$$L\phi(x) = (a(x)\nabla, \nabla)\phi(x) = \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) \quad (35)$$

with a positive definite diffusion matrix $a(x) = (a_{ij}(x))$.

In this case

$$L'g(x) = \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j} [a_{ij}(x)g(x)],$$

and consequently

$$L^{D(f)}g(x) = FL'F^{-1}g(x) = (-1)^d \int_{z \geq x} \sum_{i,j=1}^d \frac{\partial^2}{\partial z_i \partial z_j} \left[a_{ij}(z) \frac{\partial^d g(z)}{\partial z_1 \cdots \partial z_d} \right] dz_1 \cdots dz_d.$$

Let us integrate twice by parts the terms containing mixed derivatives and integrate once by parts the remaining terms. This yields

$$\begin{aligned} L^{D(f)} g(x) &= (-1)^{d-1} \sum_{j=1}^d \int_{\check{z}_j \geq \check{x}_j} \frac{\partial}{\partial x_j} \left[a_{jj}(\check{z}_j, x_j) \frac{\partial^d g(z)}{\partial z_1 \cdots \partial z_d} (\check{z}_j, x_j) \right] d\check{z}_j \\ &\quad + 2(-1)^d \sum_{i < j} \int_{\check{z}_{ij} \geq \check{x}_{ij}} \left[a_{ij} \frac{\partial^d g}{\partial z_1 \cdots \partial z_d} \right] (\check{z}_{ij}, x_i, x_j) d\check{z}_{ij}, \end{aligned}$$

where \check{z}_{ij} denotes the vector in \mathbf{R}^{d-2} obtained from z by deleting i th and j th coordinates, and $(\check{z}_{ij}, x_i, x_j)$ is the vector with i th and j th coordinates taken from the vector x , and other coordinates taken from the vector z . In case $d = 1$, the second sum in this expression is of course empty.

Again in general case one cannot simplify this expression essentially. However, assuming additionally that the coefficients a_{ij} depends only on the coordinates x_i, x_j (in particular, a_{ii} depends only on x_i), we have

$$\begin{aligned} L^{D(f)} g(x) &= (-1)^{d-1} \sum_{j=1}^d \int_{\check{z}_j \geq \check{x}_j} \frac{\partial}{\partial x_j} \left[a_{jj}(x_j) \frac{\partial^d g(z)}{\partial z_1 \cdots \partial z_d} (\check{z}_j, x_j) \right] d\check{z}_j \\ &\quad + 2(-1)^d \sum_{i < j} \int_{\check{z}_{ij} \geq \check{x}_{ij}} a_{ij}(x_i, x_j) \frac{\partial^d g}{\partial z_1 \cdots \partial z_d} (\check{z}_{ij}, x_i, x_j) d\check{z}_{ij}. \end{aligned}$$

Integrating by parts with respect to the variables \check{z}_j in the first sum and the variables \check{z}_{ij} in the second, yields (assuming the boundary terms at infinity vanish)

$$L^{D(f)} g(x) = \sum_{j=1}^d \frac{\partial}{\partial x_j} \left[a_{jj}(x_j) \frac{\partial g(x)}{\partial x_j} \right] + 2 \sum_{i < j} a_{ij}(x_i, x_j) \frac{\partial^2 g}{\partial x_i \partial x_j}, \quad (36)$$

or

$$L^{D(f)} g(x) = Lg(x) + \sum_{j=1}^d \frac{\partial a_{jj}(x_j)}{\partial x_j} \frac{\partial g(x)}{\partial x_j}. \quad (37)$$

Proposition 2.3. *Let L have form (35) with a positive definite diffusion matrix $a(x) = (a_{ij}(x))$ and with all $a_{ij} \in C^1(\mathbf{R}^d)$, so that L generates a Feller diffusion in \mathbf{R}^d that we denote X_t^x . If the coefficients a_{ij} depends only on the coordinates x_i, x_j , then $L^{D(f)}$ is given by (37) and it also generates a diffusion process in \mathbf{R}^d that we denote Y_t^y , and the duality relation (20) holds.*

Proof. Again formula (37) makes the statement very plausible, but to deduce (6) from (9) additional argument is of course needed. This goes as follows.

But notice first that it is sufficient to prove the statement under additional assumption that coefficients a_{ij} are infinitely smooth with all derivatives bounded (actually we need twice differentiability for the above calculation of $L^{D(f)}$ and d times differentiability for the formulas of Proposition 2.1 to make sense) and the operator L is strictly elliptic, because any L of type (35) can be approximated by the sequence of L of the same form but strictly elliptic and with smooth coefficients. Passing to the limit in the duality equation allows one to prove its validity for the general case.

Next, under this smoothness and non-degeneracy assumption, it is well known from the standard theory of diffusions (or Ito's processes) that operator (37) generates a unique Feller process such that its semigroup $T_t^{D(f)}$ preserves the space $C_\infty^2(\mathbf{R}^d)$ of twice continuously differentiable functions vanishing at infinity with all its derivatives up to order two. Hence, the Cauchy problem for the equation

$$\dot{g} = L^{D(f)} g$$

is well posed in classical sense for initial functions g_0 from $C_\infty^2(\mathbf{R}^d)$. It is then straightforward to see (9) that both functions $T_t^{D(f)} g_0$ and $F \circ T'_t \circ F^{-1} g_0$ satisfies this equation. Consequently these two functions coincide implying (6) for the semigroups T_t and $T_t^{D(f)}$, as required. \square

Thus we have shown that under appropriate assumptions the f -dual operators to the first order and diffusion operators respectively are again first order and diffusion operators respectively defining the f -dual or Pareto dual processes.

It is instructive to see which diffusions are self-dual. This is given by the following result that is a direct consequence of Propositions 2.3 and 2.2.

Proposition 2.4. *Let*

$$L\phi(x) = \sum_{i,j=1}^d a_{ij}(x_i, x_j) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) + \frac{1}{2} \sum_{j=1}^d a'_{jj}(x_j) \frac{\partial \phi}{\partial x_j}(x) \quad (38)$$

with a positive definite (possibly not strictly) diffusion matrix $a(x) = (a_{ij}(x))$ such that a_{ij} depend only on x_i, x_j and are continuously differentiable (with bounded derivatives). Then the diffusion generated by L is self-dual in the Pareto sense.

2.4 Application to other cones

Generalization of our results to orders arising from cones $C(e_1, \dots, e_d)$ can be obtained by the change of variables, though the calculations quickly become rather cumbersome. Let us consider only the simple example of the two-dimensional cone (18). The question we are going to answer is as follows: under what conditions the diffusion operator

$$Lg(x, y) = a(x, y) \frac{\partial^2 g}{\partial x^2} + 2b(x, y) \frac{\partial^2 g}{\partial x \partial y} + c(x, y) \frac{\partial^2 g}{\partial y^2} \quad (39)$$

generates a diffusion that has a dual in the sense of the order generated by C , and how the dual generator looks like. Having in mind the relation with the standard Pareto order we can expect that the coefficients should depend in certain way on two arbitrary functions of one variable and one arbitrary function of two variables. This is in fact the case as the following result shows.

Proposition 2.5. *Let L of form (39) with smooth coefficients generate a Feller diffusion X_t^x . If the coefficients have the form*

$$\begin{aligned} a(x, y) &= \alpha(x + y) + \beta(x - y) + \omega(x, y), \\ c(x, y) &= \alpha(x + y) + \beta(x - y) - \omega(x, y), \\ b(x, y) &= \alpha(x + y) - \beta(x - y) \end{aligned} \quad (40)$$

with some smooth functions α, β, ω , then X_t^x has the dual diffusion Y_t^y so that (11) holds with $M = C(e_1, e_2)$ of form (18), where Y_t^y is generated by the operator

$$L^{D(f)}g = Lg + 4(\alpha'(x+y) + \beta'(x-y))\frac{\partial g}{\partial x}(x, y) + 4(\alpha'(x+y) - \beta'(x-y))\frac{\partial g}{\partial y}(x, y). \quad (41)$$

Proof. Formulas (40) are obtained from Proposition 2.3 by rotation of coordinates, that is by change $x' = x + y, y' = x - y$. \square

2.5 Duality from Pareto order: jump processes

Let us now turn to the generators L of pure jump processes, that is

$$L\phi(x) = \int_{\mathbf{R}^d} (\phi(w) - \phi(x))\nu(x, dw) \quad (42)$$

with some bounded stochastic kernel ν . For a measure Q having a density with respect to Lebesgue measure, let us write shortly $L'q$ for the measure $L'Q$. We have

$$L'q(dz) = \int_{\mathbf{R}^d} q(x)\nu(x, dz)dx - q(z)dz \int_{\mathbf{R}^d} \nu(z, dw).$$

Consequently, relabeling the variables of integration, we have

$$\begin{aligned} F \circ L'(q) &= (-1)^d \int_{z \geq y} (L'q)(dz) \\ &= (-1)^d \int_{w \geq y} \int_{\mathbf{R}^d} q(z)\nu(z, dw) dz - (-1)^d \int_{z \geq y} \int_{\mathbf{R}^d} q(z)\nu(z, dw) dz. \end{aligned}$$

The integrals in the two terms partially cancel. Namely, we can write

$$F \circ L'(q) = (-1)^d \int q(z) \left(\mathbf{1}_{z \geq y} \left[\int_{w \geq y} \nu(z, dw) - \int \nu(z, dw) \right] + \mathbf{1}_{z \not\geq y} \int_{w \geq y} \nu(z, dw) \right) dz,$$

implying

$$F \circ L'(q) = (-1)^d \int q(z) \left[\mathbf{1}_{z \not\geq y} \int_{w \geq y} \nu(z, dw) - \mathbf{1}_{z \geq y} \int_{w \not\geq y} \nu(z, dw) \right] dz.$$

Hence, for a smooth (d times differentiable) function g we can write either

$$\begin{aligned} L^{D(f)}g &= F \circ L' \circ F^{-1}g(y) \\ &= (-1)^d \int \frac{\partial^d g(z)}{\partial z_1 \cdots \partial z_d} \left[\mathbf{1}_{z \not\geq y} \int_{w \geq y} \nu(z, dw) - \mathbf{1}_{z \geq y} \int_{w \not\geq y} \nu(z, dw) \right] dz, \end{aligned} \quad (43)$$

or

$$\begin{aligned} L^{D(f)}g &= F \circ L' \circ F^{-1}g(y) \\ &= (-1)^d \int_{w \geq y} \int_{\mathbf{R}^d} \frac{\partial^d g(z)}{\partial z_1 \cdots \partial z_d} \nu(z, dw) dz - (-1)^d \int_{z \geq y} \int_{\mathbf{R}^d} \frac{\partial^d g(z)}{\partial z_1 \cdots \partial z_d} \nu(z, dw) dz. \end{aligned} \quad (44)$$

If $\nu(z, dw)$ depends smoothly on z , this expression can be rewritten by moving the derivatives from g to ν . For this transformation expression (44) is more handy than (43). To perform the integration by parts in its second term we shall use the following simple formula (with a straightforward proof by mathematical induction)

$$\int_{z \geq y} \frac{\partial^d g(z)}{\partial z_1 \cdots \partial z_d} \phi(z) dz = (-1)^d \sum_{I \subset \{1, \dots, d\}} \int_{z_I \geq y_I} g(y_{\bar{I}}, z_I) \frac{\partial^{|I|} \phi}{\partial z_I} (y_{\bar{I}}, z_I) dz_I, \quad (45)$$

which is valid when the boundary terms at infinity vanish, for instance if either ϕ or g vanish at infinity with all its derivatives. Here $|I|$ is the number of indices in I , the integral over the set $\{z_I \geq y_I\}$ is $|I|$ -dimensional and $(y_{\bar{I}}, z_I)$ denotes the vector whose coordinates with indices from I are those of the vector z and other coordinates are from the vector y .

Using this formula we transform (44) into the expression

$$\begin{aligned} L^{D(f)} g(y) &= F \circ L' \circ F^{-1} g(y) \\ &= \int_{w \geq y} \int_{\mathbf{R}^d} g(z) \frac{\partial^d \nu}{\partial z_1 \cdots \partial z_d} (z, dw) dz - \sum_{I \subset \{1, \dots, d\}} \int_{z_I \geq y_I} dz_I g(y_{\bar{I}}, z_I) \int_{\mathbf{R}^d} \frac{\partial^{|I|} \nu}{\partial z_I} (y_{\bar{I}}, z_I, dw). \end{aligned}$$

Singling out from the sum the terms corresponding to I being empty and I being the whole set $\{1, \dots, d\}$, this rewrites as

$$\begin{aligned} &\int_{w \geq y} \int_{\mathbf{R}^d} g(z) \frac{\partial^d \nu}{\partial z_1 \cdots \partial z_d} (z, dw) dz - \int_{z \geq y} \int_{\mathbf{R}^d} g(z) \frac{\partial^d \nu}{\partial z_1 \cdots \partial z_d} (z, dw) dz \\ &- \sum'_{I \subset \{1, \dots, d\}} \int_{z_I \geq y_I} dz_I g(y_{\bar{I}}, z_I) \int_{\mathbf{R}^d} \frac{\partial^{|I|} \nu}{\partial z_I} (y_{\bar{I}}, z_I, dw) - g(y) \int_{\mathbf{R}^d} \nu(y, dw), \end{aligned}$$

where \sum' denotes the sum over all proper subsets I , i.e. all subsets I excluding empty set and the whole set $\{1, \dots, d\}$. Performing the cancelation in the first two terms yields finally (see the trick leading to (43))

$$\begin{aligned} L^{D(f)} g(y) &= F \circ L' \circ F^{-1} g(y) = -g(y) \int_{\mathbf{R}^d} \nu(y, dw) \\ &- \sum'_{I \subset \{1, \dots, d\}} \int_{z_I \geq y_I} dz_I g(y_{\bar{I}}, z_I) \int_{\mathbf{R}^d} \frac{\partial^{|I|} \nu}{\partial z_I} (y_{\bar{I}}, z_I, dw) \\ &+ \int_{\mathbf{R}^d} g(z) dz \left[\mathbf{1}_{z \neq y} \int_{w \geq y} \frac{\partial^d \nu}{\partial z_1 \cdots \partial z_d} (z, dw) - \mathbf{1}_{z \geq y} \int_{w \neq y} \frac{\partial^d \nu}{\partial z_1 \cdots \partial z_d} (z, dw) \right]. \quad (46) \end{aligned}$$

For instance, for $d = 1$

$$L^{D(f)} g(y) = \int_{-\infty}^y g(z) dz \int_{w \geq y} \frac{\partial \nu}{\partial z} (z, dw) - \int_y^\infty g(z) dz \int_{w < y} \frac{\partial \nu}{\partial z} (z, dw) - g(y) \int \nu(y, dw), \quad (47)$$

which is the formula essentially obtained in [21] and [22], and for $d = 2$

$$L^{D(f)} g(y) = -g(y_1, y_2) \int \nu(y, dw)$$

$$\begin{aligned}
& - \int_{z_1 \geq y_1} g(z_1, y_2) dz_1 \int \frac{\partial \nu}{\partial z_1}(z_1, y_2, dw) - \int_{z_2 \geq y_2} g(y_1, z_2) dz_2 \int \frac{\partial \nu}{\partial z_2}(y_1, z_2, dw) \\
& + \int g(z_1, z_2) dz_1 dz_2 \left[\mathbf{1}_{z \notin y} \int_{w \geq y} \frac{\partial^2 \nu}{\partial z_1 \partial z_2}(z, dw) - \mathbf{1}_{z \geq y} \int_{w \notin y} \frac{\partial^2 \nu}{\partial z_1 \partial z_2}(z, dw) \right]. \quad (48)
\end{aligned}$$

Remark 6. It is worth stressing that one should be cautious in using these formulas as they may not be true for f not vanishing at infinity, say even for a constant function f (so that these formulas cannot be used even for checking conservativity condition $L^{D(f)} \mathbf{1} = 0$). Generally one has to use the following extension of (45) (also proved by direct induction) that is valid whenever g, ϕ are smooth and such that for all $I \subset \{1, \dots, d\}$ and $y_{\bar{I}}$ there exist finite limits of the functions $g(y_{\bar{I}}, z_I)$, $\phi(y_{\bar{I}}, z_I)$ and their derivatives in z_I , as $z_I \rightarrow \infty$ (here ∞ means precisely $+\infty$):

$$\begin{aligned}
& \int_{z \geq y} \frac{\partial^d g(z)}{\partial z_1 \cdots \partial z_d} \phi(z) dz \\
& = (-1)^d \sum_{I \subset \{1, \dots, d\}} \int_{z_I \geq y_I} \left[\sum_{J \subset \bar{I}} (-1)^{|J|} g(y_{\bar{I} \setminus J}, \infty_J, z_I) \frac{\partial^{|I|} \phi}{\partial z_I}(y_{\bar{I} \setminus J}, \infty_J, z_I) \right] dz_I, \quad (49)
\end{aligned}$$

where $(y_{\bar{I} \setminus J}, \infty_J, z_I)$ denotes the vector with $\bar{I} \setminus J$ -coordinates from y , I -coordinates from z and other coordinates being $+\infty$. For instance, in case $d = 2$ we have

$$\begin{aligned}
& \int_{y_1}^{\infty} \int_{y_2}^{\infty} \frac{\partial^2 g(z)}{\partial z_1 \partial z_2} \phi(z) dz = \int_{y_1}^{\infty} \int_{y_2}^{\infty} \frac{\partial^2 \phi(z)}{\partial z_1 \partial z_2} g(z) dz \\
& + \int_{y_1}^{\infty} \left[g(z_1, y_2) \frac{\partial^2 \phi}{\partial z_1}(z_1, y_2) - g(z_1, \infty) \frac{\partial^2 \phi}{\partial z_1}(z_1, \infty) \right] dz_1 \\
& + \int_{y_2}^{\infty} \left[g(y_1, z_2) \frac{\partial^2 \phi}{\partial z_2}(y_1, z_2) - g(\infty, z_2) \frac{\partial^2 \phi}{\partial z_2}(\infty, z_2) \right] dz_2 \\
& + g(y_1, y_2) \phi(y_1, y_2) - g(\infty, y_2) \phi(\infty, y_2) - g(y_1, \infty) \phi(y_1, \infty) + g(\infty, \infty) \phi(\infty, \infty). \quad (50)
\end{aligned}$$

Assuming that for all y

$$\lim_{z \rightarrow -\infty} \int_{w \geq y} \nu(z, dw) = 0, \quad \lim_{z \rightarrow \infty} \int_{w < y} \nu(z, dw) = 0, \quad (51)$$

equation (47) rewrites in the equivalent conservative form

$$L^{D(f)} g(y) = \int_{-\infty}^y (g(z) - g(y)) dz \int_{w \geq y} \frac{\partial \nu}{\partial z}(z, dw) - \int_y^{\infty} (g(z) - g(y)) dz \int_{w < y} \frac{\partial \nu}{\partial z}(z, dw). \quad (52)$$

Proposition 2.6. Let L have form (42) with a bounded weakly continuous stochastic kernel ν , so that L generates a C-Feller (i.e. its semigroup preserves continuous functions) jump process in \mathbf{R}^d that we denote X_t^x . Then $L^{D(f)}$ is given by (43). If the kernel ν has continuous bounded mixed derivatives, so that

$$\frac{\partial^{|I|} \nu}{\partial z_I}(z, dw)$$

is again a bounded kernel (possibly signed) for any nonempty subset $I \in \{1, \dots, d\}$ (including the whole set $\{1, \dots, d\}$), then $L^{D(f)}$ can be rewritten as (46). Finally $L^{D(f)}$ generates itself a C -Feller Markov process that we denote Y_t^y if and only if the following conditions hold:

All mixed derivatives of orders from 1 to $d - 1$ of the jump rates are non-positive, i.e.

$$\int_{\mathbf{R}^d} \frac{\partial^{|I|}\nu}{\partial z_I}(z, dw) \leq 0 \quad (53)$$

for any proper subset I of $\{1, \dots, d\}$; and

$$\begin{aligned} \int_{w \geq y} \frac{\partial^d \nu}{\partial z_1 \cdots \partial z_d}(z, dw) &\geq 0, \quad z \not\geq y, \\ \int_{w \not\geq y} \frac{\partial^d \nu}{\partial z_1 \cdots \partial z_d}(z, dw) &\leq 0, \quad z \geq y. \end{aligned} \quad (54)$$

If this is the case, the duality relation (20) holds.

Proof. Everything is proved apart from the criterion for the generation of a Markov process. To get it one only has to note that the operator $\int g(z)\mu(y, dz) - \alpha(y)g(y)$ with given kernel μ and function α is conditionally positive (and generates a process) if and only if the kernel μ is stochastic (i.e. positive), and that the kernels from various terms in (46) are mutually singular, so that this positivity condition should be applied separately to each term.

One completes the proof by the same argument as used at the end of the proof of Proposition 2.3. \square

Couple of remarks are in order here. Condition (54) is not very transparent. A simple particular case to have in mind is when the kernel ν decomposes into a sum of kernels depending on all variables but for one, i.e.

$$\nu(z, dw) = \sum_j \nu_j(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_d, dw),$$

in which case the condition (54) becomes void (thus trivially satisfied). On the other hand, conditions (53) are easy to check. To visualize this condition it is instructive to observe that if q is a density of a positive measure on \mathbf{R}^d , then the distribution function

$$g(x) = \int_{z \neq x} q(z) dz$$

is positive, but has all mixed derivatives negative. Even more specifically, if ν decomposes into a sum of kernels depending on one variable only, that is

$$\nu(z, dw) = \sum_j \nu_j(z_j, dw),$$

all conditions of Proposition 2.6 are reduced to an easy to check requirement that all rates $\int \nu_j(z_j, dw)$ are decreasing functions of z_j .

Let us note that the method of the calculation of dual used above can still be used for processes with a boundary. For instance, let us consider a process on \mathbf{R}_+ with the generator

$$L\phi(x) = \int_{\mathbf{R}_+} (\phi(w) - \phi(x))\nu(x, dw). \quad (55)$$

The operator L' takes the form

$$L'q(dz) = \int_{\mathbf{R}_+} q(x)\nu(x, dz)dx - q(z)dz \int_{\mathbf{R}_+} \nu(z, dw)$$

and the same calculations as above yield

$$\begin{aligned} L^{D(f)}g(y) &= \int_0^y g(z) dz \int_{w \geq y} \frac{\partial \nu}{\partial z}(z, dw) - \int_y^\infty g(z) dz \int_{0 \leq w < y} \frac{\partial \nu}{\partial z}(z, dw) \\ &\quad - g(y) \int \nu(y, dw) + g(0) \int_{w \geq y} \nu(0, dw), \end{aligned} \quad (56)$$

that is, an additional term appears arising from additional boundary taken into account while integrating by parts. Under assumption (51), this rewrites in the equivalent conservative form

$$\begin{aligned} L^{D(f)}g(y) &= \int_0^y (g(z) - g(y)) dz \int_{w \geq y} \frac{\partial \nu}{\partial z}(z, dw) \\ &\quad - \int_y^\infty (g(z) - g(y)) dz \int_{0 \leq w < y} \frac{\partial \nu}{\partial z}(z, dw) + \int_{w \geq y} (g(0) - g(y))\nu(0, dw). \end{aligned} \quad (57)$$

We assume strong smoothness condition for ν , which forces the dual Lévy kernel to have a density. This is not necessary. Just assuming monotonicity of $\int_{w \geq y} \nu(z, dw)$ and $\int_{w < y} \nu(z, dw)$ (and thus the existence almost sure of non-negative derivatives of these functions of z), we obtain, instead of (58), the formula

$$L^{D(f)}g(y) = \int_{-\infty}^y (g(z) - g(y)) dz \int_{w \geq y} \nu(z, dw) - \int_y^\infty (g(z) - g(y)) dz \int_{w < y} \nu(z, dw), \quad (58)$$

with similar modifications for (57) and analogously for d -dimensional case.

Let us mention the link with the theory of stochastic monotonicity. A Markov process X_t^x is called stochastically monotone with respect to Pareto ordering if the function $\mathbf{P}(X_t^x \geq y)$ is a monotone function of x for any y . Stochastic monotonicity is studied for various classes of processes, see [9], [10], [21], [23], [33], [32], [28] and references therein. If duality (20) holds, then X_t^x is obviously stochastically monotone, but, generally speaking, this condition is too weak to ensure duality, because stochastic monotonicity of a positive function on \mathbf{R}^d does not imply (apart from one-dimensional case) that it is the multi-dimensional distribution function for some positive measure. Therefore it is remarkable enough that for diffusion processes with generators (35) the conditions of stochastic monotonicity and of the existence of Pareto dual coincide. Even for deterministic processes this is already not so, as for stochastic monotonicity of processes generated by operators (29), b_j are allowed to depend on other coordinates x_k (in a monotone way, see e.g. [9] and references therein to previous works). Stochastic monotonicity and related duality are

well developed for Markov chains, see e.g. [2] and [31], for birth and death processes and one-dimensional diffusions see [11].

We assumed boundedness of all coefficients involved. This simplification leads to the most straightforward formulations that catch up the essence of duality. Of course, extensions to unbounded kernel rates, diffusion coefficients, etc, are possible under the conditions that ensure that all processes involved are well defined.

2.6 Arbitrary Feller processes

We have analyzed three classes of the generators L separately. But it is clear that if we consider a process with the generator being the sum of the generators of different classes, then applying conditions of the results above to each term separately will ensure that the dual to the sum is also conditionally positive and generates a process leading to the duality relation (20). For simplicity, we shall give the corresponding result for one-dimensional Feller processes, but extension to higher dimensions is straightforward. For this case, the generators of the dual were obtained in [22] (which contains an annoying systematic typo with the wrong sign ' $-$ ' before the second term of (37)) by approximating continuous state space generators by discrete Markov chains. The method of the present paper will give the same result without any technical restrictions used in [22] this yielding the complete characterization.

Proposition 2.7. *Let a Feller process X_t^x in $C_\infty(\mathbf{R})$ have a generator*

$$Lg(x) = a(x) \frac{d^2}{dx^2} g(x) + b(x) \frac{d}{dx} g(x) + \int_{-\infty}^{\infty} (g(z) - g(x) - (z-x)g'(x)\mathbf{1}_{|z-x|\leq 1})\nu(x, dz) \quad (59)$$

with $a, b \in C^2(\mathbf{R})$, a being non-negative, and with the weakly continuous Lévy kernel ν such that, for any y , conditions (51) hold and the functions

$$\int_{w \geq y} \nu(z, dw), \quad - \int_{w < y} \nu(z, dw) \quad (60)$$

are non-decreasing in z , for $z < y$ and $z > y$ respectively, so that their derivatives exist almost surely and are non-negative. Moreover

$$\mathbf{1}_{z < y} d_z \int_{w \geq y} \nu(z, dw) + \mathbf{1}_{z > y} d_z \int_{w < y} \nu(z, dw) \quad (61)$$

is a Lévy kernel (it integrates $\min(1, (w-z)^2)$) and the integral

$$\int_{y-1}^{y+1} (z-y) \left[\mathbf{1}_{z < y} (\nu(y, dz) + d_z \int_{w \geq y} \nu(z, dw)) + \mathbf{1}_{z > y} (\nu(y, dz) - d_z \int_{w < y} \nu(z, dw)) \right]$$

exists, at least in the sense of the main (or the Cauchy) value. Then the dual process Y_t^y exists (in the sense of (20)) and has the generator

$$\begin{aligned} Lg(y) &= a(y) \frac{d^2}{dy^2} g(y) + (a'(y) - b(y)) \frac{d}{dy} g(y) \\ &+ \int_{-\infty}^y (g(z) - g(y) - (z-y)g'(y)\mathbf{1}_{|z-y|\leq 1}) d_z \left(\int_{w \geq y} \nu(z, dw) \right) \end{aligned}$$

$$\begin{aligned}
& - \int_y^\infty (g(z) - g(y) - (z - y)g'(y)\mathbf{1}_{|z-y|\leq 1})d_z \left(\int_{w < y} \nu(z, dw) \right) \\
& + g'(y) \int_{y-1}^{y+1} (z-y) \left[\mathbf{1}_{z < y}(\nu(y, dz) + d_z \int_{w \geq y} \nu(z, dw)) + \mathbf{1}_{z > y}(\nu(y, dz) - d_z \int_{w < y} \nu(z, dw)) \right]
\end{aligned} \tag{62}$$

Proof. Formula (62) is obtained by combining (58), (37) and (31). Conditions given ensure that the dual operator is well defined as a Lévy-Khintchin type operator with variable coefficients. \square

Remark 7. As shown in [22] and Theorem 5.9.2 of [23], conditions of stochastic monotonicity (monotonicity of functions (60)) are sufficient for the operator (62) to generate a Feller process, so that this condition can be dispensed with.

As a corollary of Proposition 2.7, we can get now the full characterization of self-duality.

Proposition 2.8. Let a Feller process X_t^x in $C_\infty(\mathbf{R})$ have a generator (62). Then it is self dual (in the sense of (20)) if and only if the following conditions holds:

$$b(x) = a'(x)/2, \quad d_y\nu(y, dz) + d_z\nu(z, dy) = 0. \tag{63}$$

In particular, if ν has a density $\nu(z, w)$, which is differentiable with respect to the first argument, then the second equation of (63) rewrites as

$$\frac{\partial \nu}{\partial y}(y, z) + \frac{\partial \nu}{\partial z}(z, y) = 0. \tag{64}$$

Clearly, this condition is satisfied for $\nu(y, z) = g(|y - z|)$ with a smooth g , which corresponds to symmetric Lévy generators.

Proof. The condition on b follows from Proposition 2.4. The condition on ν arises by the comparison of the integral terms of (62) with (62) separately for $y > z$ and $y < z$. \square

3 Stochastic f -duality from translation invariant f

We have analyzed in some detail the duality arising from Pareto ordering. In general case explicit calculations are not always available. However, we shall propose here some general scheme for the analysis of translation-invariant f , that is f depending only on the difference of their arguments:

$$f(x, y) = f(y - x),$$

with some other function f that we still denote by f (with some ambiguity).

Thus the operator F from (5) when applied to a measure Q with density q takes the form

$$g(y) = (FQ)(y) = \int_{\mathbf{R}^d} f(y - x)q(dx), \tag{65}$$

i.e. it becomes a convolution operator. It is then well known that under appropriate regularity assumptions, f is the fundamental solution of the pseudo-differential operator L_f with the symbol

$$L_f(p) = \frac{1}{\hat{f}(p)}, \quad (66)$$

where

$$\hat{f}(p) = \int e^{-ixp} f(x) dx$$

is the Fourier transform of f .

Remark 8. *In fact, by the definition of the fundamental solution,*

$$L_f \left(\frac{1}{i} \frac{\partial}{\partial x} \right) f(x) = \delta(x),$$

which by taking the Fourier transform from both sides rewrites as

$$L_f(p) \hat{f}(p) = 1,$$

as claimed.

Hence $g(y)$ from (65) solves the equation $L_f g = q$, so that $F^{-1} = L_f$. Of course, for an arbitrary f , the operator L_f can be quite awkward and the identification of the appropriate classes of functions q, g quite nontrivial. Let us consider the case when everything is well understood, namely the case of L_f being a Laplacian, or more generally, its fractional power.

It is well known that the fundamental solution for the Laplace operator Δ in dimension $d \geq 3$ is the function

$$f(x) = -\frac{1}{(d-2)\sigma_{d-1}} \frac{1}{|x|^{d-2}},$$

where σ_{d-1} is the area of the unit sphere in \mathbf{R}^d . Hence the dual operator (6) takes the form

$$T^{D(f)} = \Delta^{-1} \circ T' \circ \Delta, \quad (67)$$

and the generator for the corresponding dual semigroup becomes

$$L^{D(f)} = \Delta^{-1} \circ L' \circ \Delta. \quad (68)$$

Let L be a diffusion operator of the special kind:

$$Lg(x) = a(x) \Delta g(x)$$

with a nonnegative bounded smooth function $a(x)$. Then $L' = \Delta \circ a(x)$ and thus

$$L^{D(f)} = \Delta^{-1} \circ L' \circ \Delta = L, \quad (69)$$

so that L is self f -dual.

Noting that in dimensions $d = 2$ the fundamental solution for the Laplacian is known to be $\log|x|/2\pi$ we get the following.

Proposition 3.1. Let X_t^x be the Feller diffusion generated by the operator $Lg(x) = a(x)\Delta g(x)$ in \mathbf{R}^d with a nonnegative bounded smooth function $a(x)$. Then, for all $x, y \in \mathbf{R}^d$, we have

$$\mathbf{E} \frac{1}{|X_t^x - y|^{d-2}} = \mathbf{E} \frac{1}{|X_t^y - x|^{d-2}}, \quad (70)$$

$$\mathbf{E} \log |X_t^x - y| = \mathbf{E} \log |X_t^y - x|, \quad d = 2 \quad (71)$$

for $d \geq 3$ and $d = 2$ respectively.

Turning to the fractional Laplacian $|\Delta|^{\alpha/2}$ in \mathbf{R}^d with $\alpha \in (0, 2)$, $d \geq 2$, let us recall that the inverse operator is given by the so-called Riesz potential

$$|\Delta|^{-\alpha/2}g(x) = I^\alpha g(x) = \frac{1}{H_d(\alpha)} \int_{\mathbf{R}^d} \frac{g(y) dy}{|x - y|^{d-\alpha}},$$

where

$$H_d(\alpha) = 2^\alpha \pi^{d/2} \frac{\Gamma(\alpha/2)}{\Gamma((d-\alpha)/2)},$$

see e.g. [17]. Hence, the operator $|\Delta|^{\alpha/2}$ is L_f for

$$f(x) = \frac{1}{H_d(\alpha)} \frac{1}{|x|^{d-\alpha}}.$$

Let us consider a stable-like process generated by the operator

$$Lg(x) = -a(x)|\Delta|^{\alpha/2}g(x)$$

with a positive smooth function $a(x)$. Then $L' = |\Delta|^{\alpha/2} \circ a(x)$ and thus

$$L^{D(f)} = |\Delta|^{-\alpha/2} \circ L' \circ |\Delta|^{\alpha/2} = L, \quad (72)$$

so that L is self f -dual. Thus we proved the following extension of Proposition 3.1:

Proposition 3.2. Let X_t^x be the stable-like process generated by the operator $Lg(x) = a(x)|\Delta|^{\alpha/2}g(x)$ in \mathbf{R}^d with $d \geq 2$, $\alpha \in (0, 2]$ excluding the case $d = \alpha = 2$ (for which (71) holds), and with a nonnegative bounded smooth function $a(x)$. Then, for all $x, y \in \mathbf{R}^d$,

$$\mathbf{E} \frac{1}{|X_t^x - y|^{d-\alpha}} = \mathbf{E} \frac{1}{|X_t^y - x|^{d-\alpha}}. \quad (73)$$

4 Stochastic duality for processes in $\bar{\mathbf{R}}_+$

4.1 Reflected and absorbed diffusions in $\bar{\mathbf{R}}_+$

We shall deduce some consequences from our general approach to processes on \mathbf{R}_+ that are dual in the sense (3). $C_\infty^k(\mathbf{R}^d)$ will denote the space of k times differentiable functions on \mathbf{R}^d with all these derivatives vanishing at infinity. $C_\infty^k(\bar{\mathbf{R}}_+)$ is the restriction of functions from $C_\infty^k(\mathbf{R})$ on $\bar{\mathbf{R}}_+ = \{x \geq 0\}$.

Consider a Feller process X_t^x on \mathbf{R} generated by operator (62) under the conditions of Proposition 2.7 assuming additionally that

(A) $a \in C^2(\mathbf{R})$ and is an even function such that $a(x) \geq 0$, $b \in C^2(\mathbf{R})$ and is an odd function (implying $b(0) = 0$), the support of ν is in \mathbf{R}_+ for $x \geq 0$ and $\nu(-x, dy) = R\nu(x, dy)$, where R denotes the reflection of the measure with respect to the origin (so that, by definition, $\int \phi(y)R\nu(x, dy) = \int \phi(-y)\nu(x, dy)$).

Then, as is well known, see e.g. Theorem 6.8.1 in [23], the magnitude $|X_t^x|$ is itself a Markov process on \mathbf{R}_+ , also referred to as X_t^x reflected at the origin. Moreover, if the transition probabilities of X_t^x are $p_t(x, dy)$, then $|X_t^x|$ has the transition density

$$p_t^{ref}(x, dy) = p_t(x, dy) + Rp_t(x, dy),$$

and the semigroup T_t^{ref} of $|X_t^x|$ can be obtained from the semigroup T_t of X_t^x by the restriction to even functions.

Remark 9. (i) Assuming that the kernel ν is twice smooth would imply that the space $C_\infty^2(\mathbf{R})$ is an invariant core for X_t^x and consequently that the subspace of functions f from $C_\infty^2(\bar{\mathbf{R}}_+)$ such that $f'(0) = 0$ is an invariant core for $|X_t^x|$. (ii) If X_t^x were a diffusion, the process $|X_t^x|$ on $\bar{\mathbf{R}}_+$ would be stochastically monotone by the coupling argument, see e.g. Sect II,2 of [25]) and hence by Siegmund's theorem [29] it had a Markov dual Y_t^y on $\bar{\mathbf{R}}_+$ (in the sense (3)) with absorption at the origin. In our case monotonicity follows from the construction of the dual below, which turns out to be given by a semigroup with a conditionally positive generator.

Proposition 4.1. Under the conditions of Proposition 2.7, assumption (A) above and finally assuming that the measure $\nu(0, dw)$ is bounded, the dual process Y_t^y is a Feller on $\bar{\mathbf{R}}_+$ absorbed at the origin and generated by the operator

$$\begin{aligned} L^D g(y) &= a(y) \frac{d^2}{dy^2} g(y) + (a'(y) - b(y)) \frac{d}{dy} g(y) + \int_{w \geq y} (g(0) - g(y)) \nu(0, dw) \\ &\quad + \int_0^y (g(z) - g(y) - (z - y)g'(y) \mathbf{1}_{|z-y| \leq 1}) d_z \left(\int_{w \geq y} \nu(z, dw) \right) \\ &\quad - \int_y^\infty (g(z) - g(y) - (z - y)g'(y) \mathbf{1}_{|z-y| \leq 1}) d_z \left(\int_{w < y} \nu(z, dw) \right) \\ &+ g'(y) \int_{y-1}^{y+1} (z-y) \left[\mathbf{1}_{z < y} (\nu(y, dz) + d_z \int_{w \geq y} \nu(z, dw)) + \mathbf{1}_{z > y} (\nu(y, dz) - d_z \int_{w < y} \nu(z, dw)) \right] \end{aligned} \tag{74}$$

The semigroup T_t^D of Y_t^y is given explicitly by the formula

$$(T_t^D g)(y) = g(0) \int_y^\infty p_t^{ref}(0, dz) + \int_0^\infty g(x) \left(\int_y^\infty \frac{\partial}{\partial x} p_t^{ref}(x, dz) \right) dx. \tag{75}$$

Proof. Using (8) with $F^{-1}g(x) = -g'(x)$ we get for $g \in C_\infty^1(\bar{\mathbf{R}}_+)$

$$(T_t^D g)(y) = - \int_y^\infty dz \int_0^\infty g'(x) p_t^{ref}(x, dz) dx, \tag{76}$$

and hence

$$(T_t^D g)(y) = g(0) \int_y^\infty p_t^{ref}(0, dz) + \int_0^\infty dx \int_y^\infty g(x) \frac{\partial}{\partial x} p_t^{ref}(x, dz), \tag{77}$$

yielding (75) as required.

It is worth stressing that this formula implies the conservativity condition $T_t^D \mathbf{1} = \mathbf{1}$ (preservation of constants by T_t^D), because

$$\lim_{x \rightarrow \infty} \int_y^\infty p_t^{ref}(x, dz) = 1$$

by the Feller property and hence

$$\int_0^\infty \frac{\partial}{\partial x} \left(\int_y^\infty p_t^{ref}(x, dz) \right) dx = 1 - \int_y^\infty p_t^{ref}(0, dz). \quad (78)$$

Operators T_t^D form a semigroup by Proposition 1.2. The form of the generator follows from (57). As it is conditionally positive, the semigroup T_t^D preserves positivity and preserves constants thus being a semigroup of a Markov process. Moreover, as also seen directly from (75), $T_t^D f(0) = f(0)$, so that the value at the origin is preserved meaning that this process is absorbing at the origin. \square

Remark 10. (i) Formula (76) is valid only for g vanishing at infinity, and (75) extends it (yields a minimal extension) to bounded functions on $\bar{\mathbf{R}}_+$. Plugging $g = 1$ into (76) yields zero, not 1. (ii) The attempt to use integration in (78) in opposite direction, at least when $p_t(x, dz)$ has a density $p_t(x, z)$, and using $\lim_{x \rightarrow \infty} p_t^{ref}(x, z) = 0$ would give

$$\int_y^\infty dz \left(\int_0^\infty \frac{\partial}{\partial x} p_t^{ref}(x, z) dx \right) = - \int_y^\infty p_t^{ref}(0, z) dz,$$

which is different from the r.h.s. of (78).

It is worth noting additionally that if $a(0) \neq 0$ and $\nu = 0$, then the subspace of functions g from $C_\infty^2(\bar{\mathbf{R}}_+)$ such that $g''(0) = 0$ is an invariant core for Y_t^y . In fact, the condition $L^D g(0) = 0$ (following from $T_t^D g(0) = g(0)$) implies $g''(0) = 0$. On the other hand, if $a(0) = 0$ and $\nu = 0$, then $a(x) = ax^2(1 + o(1))$, $b(x) = bx(1 + o(1))$ as $x \rightarrow 0$ with $a \geq 0$, $b \in \mathbf{R}$ implying that 0 is an unaccessible boundary point, so that $X_t^x = |X_t^x|$ for $x > 0$. In this case nothing comes out of the origin, so that $p_t^{ref}(0, z) = 0$ for all $z > 0$ implying that the first term on the r.h.s. of (75) vanishes and hence that 0 is also unaccessible for Y_t^y (which follows also from its generator). In particular, if additionally $b(x) = a'(x)/2$, the process $|X_t^x|$ is self-dual on \mathbf{R}_+ .

There is an extensive literature on the absorption - reflection link presented in Proposition 4.1, mostly because of its natural interpretation in terms of ruin probabilities having important applications in insurance mathematics. For piecewise deterministic Markov processes it was obtained in [6] (see also [4]) and used effectively in [12] for assessing ruin probabilities via large deviations. Then it was extended to diffusions with jumps in [30], and to Lévy processes in [5]. Our result is an extension of the corresponding result from [30], as we do it for arbitrary stochastically monotone processes. Our proof is quite different, as it is more elementary, using effectively only formula (8).

4.2 Second dual and regularized dual

Extension of the previous result to processes with a boundary from the right or with two boundaries is of course natural, see [5], but not quite straightforward. We shall clarify the

aspects of duality (even the definition has to be modified), needed for these cases reducing our attention to diffusions just for simplicity

It is natural to ask whether the second dual coincides with the original process. For diffusions on \mathbf{R}^d this is in fact the case, as is seen from Proposition 2.3. However, for processes on \mathbf{R}_+ this does not hold, as seen already from Lévy's example of reflected Brownian motion. In fact, reflected BM cannot be dual to absorbing BM, as any dual process on \mathbf{R}_+ should be absorbing at the left end, that is at the origin, as seen directly from (3). However, the reflected BM is 'almost dual' to the absorbing BM in the sense that $\mathbf{P}(Y_t^y \leq x) = \mathbf{P}(X_t^x \geq y)$ (with Y reflected and X absorbing BM) holds for all $y \neq 0$ and all x . This suggests that the usual definition of duality imposes unnatural restrictions on the boundary. Consequently we shall give the following definition. Let X_t^x be a stochastically monotone process on $[a, \infty)$ such that $\mathbf{P}(X_t^x \geq y)$ is right continuous in x . A process Y_t^y on $[a, \infty)$ will be called a *regularized dual* to a process X_t^x on $[a, \infty)$ if (3) holds for all $x \geq a, y > a$, and the distribution for $y = a$ is defined by continuity as

$$\mathbf{P}(Y_t^a \leq x) = \mathbf{P}(Y_t^{a-} \leq x) = \lim_{y \rightarrow a} \mathbf{P}(Y_t^y \leq x). \quad (79)$$

Remark 11. (i) One could also relax the condition for $x = a$ defining $\mathbf{P}(Y_t^y \leq a) = \lim_{x \rightarrow a} \mathbf{P}(Y_t^y \leq x)$, but this would lead to the same result, as for usual definition, due to the right continuity of $\mathbf{P}(X_t^x \geq y)$ in x . (ii) If one only assumes monotonicity of the function $\mathbf{P}(X_t^x \geq y)$, it would become natural to define the dual distribution $\mathbf{P}(Y_t^y \leq x)$ as the right continuous modification of the function $\mathbf{P}(X_t^x \geq y)$.

The following statement is now clear.

Proposition 4.2. Under the assumptions of Proposition 4.1 the initial reflected process $|X_t^x|$ is a regularized dual to Y_t^y . Thus the second regularized dual to $|X_t^x|$ coincides with $|X_t^x|$.

Remark 12. The usual (not regularized) dual of Y_t^x from Proposition 4.1 is a rather pathological process Z_t^z , whose distributions coincides with that of $|X_t^z|$ for $z \neq 0$, but the origin is an unattainable point without escape from it. Thus Z_t^z should be 'reflected from the origin' without touching it.

Remark 13. Of course one can deal with reflected processes on $\bar{\mathbf{R}}_-$ by introducing a symmetric notion of duality. Namely, for a process X_t^x on an interval of \mathbf{R} let us say that Y_t^y is its right dual, if $\mathbf{P}(Y_t^y \leq x) = \mathbf{P}(X_t^x \geq y)$ holds for all x, y (that is, it is the usual duality used above) and left dual if $\mathbf{P}(Y_t^y < x) = \mathbf{P}(X_t^x > y)$ holds for all x, y , which is equivalent to $\mathbf{P}(Y_t^y \geq x) = \mathbf{P}(X_t^x \leq y)$. Thus, by definition, Y_t^y is right dual to X_t^x if and only if X_t^x is left dual to Y_t^y . The theory of left dual processes on \mathbf{R}_- (and their regularized version) is completely analogous to the theory of right dual process on \mathbf{R}_+ .

References

- [1] R. Alkemper, M. Hutzenthaler. Graphical representation of some duality relations in stochastic population models. Electron. Comm. Probab. 12 (2007), 206220 (electronic).

- [2] W.J. Anderson. Continuous -Time Markov Chains. Probability and its Applications. Springer Series in Statistics. Springer 1991.
- [3] L. Angiuli, D. Pallara, F. Paronetto. Analytic semigroups generated in $L^1(\Omega)$ by second order elliptic operators via duality methods. Semigroup Forum 80:2 (2010), 255-271.
- [4] S. Asmussen. Subexponential Asymptotics for Stochastic Processes: Extremal Behavior, Stationary Distributions and First Passage Probabilities. Ann. Appl. Prob. 8:2 (1998), 354-374.
- [5] S. Asmussen and M. Pihlsgard. Loss Rates for Lévy Processes with Two Reflecting Barriers. Math. Oper. Res. 32:2 (2007), 308-321.
- [6] S. Asmussen and S. S. Petersen. Ruin Probabilities Expressed in Terms of Storage Processes. Adv. Appl. Prob. 20:4 (1988), 913-916.
- [7] Ph. Biane. Intertwining of Markov Semi-Groups, some Examples. In: Séminaire de probabilités XXIX, Lectre Notes in Mathematics v. 1613, Springer 1995, p. 30-36.
- [8] P. Carmona, F. Petit and M. Yor. Beta-gamma random variables and intertwining relations between certain Markov processes. Rev. Mat. Iberoamericana 14:2 (1998), 311-367.
- [9] M. F. Chen, F.Y. Wang. On order-preservation and positive correlations for multi-dimensional diffusion process. Prob. Th. Rel. Fields 95 (1993), 421-428.
- [10] M. F. Chen. From Markov Chains to Non-Equilibrium Particle Systems, World Scientific, Singapore, 2nd Ed. 2004.
- [11] J. Th. Cox, U. Rösler. A duality relation for entrance and exit laws for Markov processes. Stochastic Processes and their Applications 16 (1983), 141- 156.
- [12] B. Djehiche. A Large Deviation Estimate for Ruin Probabilites. Scand. Actuarial J. 1 (1993), 42-59.
- [13] J. Dubédat. Reflected planar Brownian motions, intertwining relations and crossing probabilities. Ann. Inst. H. Poincar Probab. Statist. 40:5 (2004), 539-552.
- [14] A.M. Etheridge. An Introduction to Superprocesses. University Lecture Series 20, AMS, Providence, RI, 2000.
- [15] S.N. Ethier, Th. G. Kurtz. Markov Processes – Characterization and Convergence. Wiley Series in Probability and Mathematical Statistics, New York Chichester: Wiley 1986.
- [16] I.M. Gelfand and G. E. Shilov. Generalized Functions. V.1: Properties and Operations. Transl. from Russian. Academic Press 1964.
- [17] S. Helgason. The Radon transform. Progress in Mathematics, 5. Birkhäuser, Boston, Mass., 1980. Sec Ed. 1999.

- [18] F. Hirsch, M. Yor. Fractional intertwinings between two Markov semigroups. *Potential Anal.* 31:2 (2009), 133-146.
- [19] Th. Huillet, S. Martinez. Duality and intertwining for discrete Markov kernels: relations and examples. *Adv. in Appl. Probab.* 43:2 (2011), 437-460.
- [20] O. Kallenberg. *Foundations of Modern Probability*. Second ed., Springer 2002.
- [21] V.N. Kolokoltsov. Measure-valued limits of interacting particle systems with k -nary interactions I. One-dimensional limits. *Probability Theory Related Fields* **126** (2003), 364-394.
- [22] V. N. Kolokoltsov. Stochastic monotonicity and duality for one-dimensional Markov processes. arXiv:1002.4773 (2010). *Mathematical Notes* **89:5** (2011), 652-660.
- [23] V. N. Kolokoltsov. *Markov processes, semigroups and generators*. DeGruyter Studies in Mathematics v. 38, DeGruyter, 2011.
- [24] V. Kolokoltsov. On stochastic duality for time nonhomogeneous Markov processes. Preprint 2013.
- [25] Th. Liggett. *Interacting Particle Systems*. Springer, 2005.
- [26] L. Mytnik. Superprocesses in random environments. *Ann. Prob.* **24** (1996), 1953-1978.
- [27] P. Patie, Th. Simon. Intertwining certain fractional derivatives. *Potential Anal.* 36:4 (2012), 569-587.
- [28] L. Rabehasaina. Monotonicity properties of multi-dimensional reflected diffusions in random environment and applications. *Stochastic processes and their Applications* **116** (2006), 178-199.
- [29] D. Siegmund. The equivalence of absorbing and reflecting barrier problems for stochastically monotone Markov processes. *Ann. Probability* 4:6 (1976), 914-924.
- [30] K. Sigman and R. Ryan. Continuous-Time Monotone Stochastic Recursions and Duality. *Adv. Appl. Prob.* **32:2** (2000), 426-445.
- [31] E. A. Van Doorn. Stochastic monotonicity of birth-death processes. *Adv. in Appl. Probab.* 12:1 (1980), 5980.
- [32] J.-M. Wang. Stochastic Comparison and Preservation of Positive Correlations for Lévy-type processes. *Acta Math. Sinica, English Series*, 25(2009), 741-758.
- [33] Y.H. Zhang Y. H. Sufficient and necessary conditions for stochastic comparability of jump processes. *Acta Mathematica Sinica, English Series*, 2000, **16:1** (2000), 99-102.